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INTEGRALS OF RELATIVE MOTION OF A SYSTEM OF PARTICLES OF IDENTICAL MASS

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It is shown that the Hamiltonian function of a closed system of particles of identical mass and its moment of impulse are sums of the corresponding integrals of motion of the center of inertia and of $A(A-1)/2$ integrals of relative motion where A is the number of particles.

Using the laws of conservation, we can separate the independent variables of the multi-particle equations of motion only in particular cases revealed by Euler and Lagrange [1]. To find other such cases, it is expedient to describe the internal motion of the closed multi-particle systems in terms of the relative coordinates, which for the three-particle systems are defined by the relations [2]

$$\eta_1 = r_2 - r_1, \qquad \eta_2 = r_3 - r_2, \qquad \eta_3 = r_1 - r_3 \tag{1}$$
$$\eta_1 + \eta_2 + \eta_3 \equiv 0$$

where r_k are independent radius vectors of the particles. Then, provided that the relative variables are chosen to be equivalent

$$-\eta_i = \eta_j + \eta_k, \qquad -\eta_j = \eta_i + \eta_k, \qquad -\eta_k = \eta_i + \eta_j \tag{2}$$

we can postulate that upon differentiation only two vectors in the triangle (1) can vary simultaneously, and the following theorem holds.

Theorem. The condition that the interacting particles are all of mass A is both necessary and sufficient for the Hamiltonian function of the system and its moment of impulse to be sums of the corresponding integrals of motion of the center of inertia and of $A(A-1)/2$ integrals of relative motion.

1. Three-particle systems.

Proof. Using Eqs. (1) and the relation

$$M\mathbf{R} = \sum_{k=1}^{3} m_k \mathbf{r}_k, \qquad M = \sum_{k=1}^{3} m_k \tag{1.1}$$

defining the radius vector R of the center of inertia, we connect the independent vectors with the relative coordinates by means of expressions

$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2}{M}\eta_1 + \frac{m_3}{M}\eta_3, \quad \mathbf{r}_2 = \mathbf{R} + \frac{m_1}{M}\eta_1 - \frac{m_3}{M}\eta_2, \quad \mathbf{r}_3 = \mathbf{R} + \frac{m_2}{M}\eta_2 - \frac{m_3}{M}\eta_3 \tag{1.2}$$

which for the equal masses are simplified as follows:

$$\mathbf{r}_1 = \mathbf{R} + (-\eta_1 + \eta_3)/3, \quad \mathbf{r}_2 = \mathbf{R} + (\eta_1 - \eta_2)/3, \quad \mathbf{r}_3 = \mathbf{R} + (\eta_2 - \eta_3)/3 \tag{1.3}$$

Taking the conditions of equivalence (2) into account, we obtain

$$\left.\frac{\partial \eta_i}{\partial \eta_j}\right|_{\eta_k} = 1 \quad (i = j \neq k), \qquad \left.\frac{\partial \eta_i}{\partial \eta_j}\right|_{\eta_k} = -1 \quad (i \neq j \neq k) \tag{1.4}$$

Applying the rules of differentiation of compound functions and using (1.4), we find the relative gradients and obtain the following operator identity:

$$\nabla_{\eta_1} = (\nabla_{r_2} - \nabla_{r_1})/3, \quad \nabla_{\eta_2} = (\nabla_{r_3} - \nabla_{r_2})/3, \quad \nabla_{\eta_3} = (\nabla_{r_1} - \nabla_{r_3})/3 \qquad (1.5)$$
$$\nabla_{\eta_1} + \nabla_{\eta_2} + \nabla_{\eta_3} \equiv 0$$

The latter is then applied to the sum of functions $\varphi_1(\eta_1) + \varphi_2(\eta_2) + \varphi_3(\eta_3)$ possessing first order derivatives, to obtain the following expression which is essential to our proof:

$$\sum_{i=1}^{3} \nabla_{\eta_i} \sum_{j=1}^{3} \varphi_j(\eta_j) = (\nabla_{\eta_1}\varphi_1) + (\nabla_{\eta_2}\varphi_2)\frac{\partial\eta_2}{\partial\eta_1} + \ldots + (\nabla_{\eta_3}\varphi_3) = -\sum_{i=1}^{3} (\nabla_{\eta_i}\varphi_i) \equiv 0 \qquad (1.6)$$

Indeed, differentiating with respect to time and multiplying the relations (1.2) by the corresponding masses m_k we connect the particle impulses with the relative impulses $q_j = \mu_j\eta_j$ (where μ_j denote the effective masses [2]) and with the center of inertia impulse $\mathbf{P}$

$$p_1 = \frac{m_1}{M}\mathbf{P} - q_1 + q_3, \quad p_2 = \frac{m_2}{M}\mathbf{P} + q_1 - q_2, \quad p_3 = \frac{m_3}{M}\mathbf{P} + q_2 - q_3 \qquad (1.7)$$

Using Eqs. (1) we can express the relative impulses in terms of the particle impulses, connecting them by the following identity

$$q_1 = \frac{m_1}{M}p_2 - \frac{m_2}{M}p_1, \quad q_2 = \frac{m_2}{M}p_3 - \frac{m_3}{M}p_2, \quad q_3 = \frac{m_3}{M}p_1 - \frac{m_1}{M}p_3, \qquad (1.8)$$
$$m_3q_1 + m_1q_2 + m_2q_3 \equiv 0$$

Next we obtain the Hamiltonian and Lagrangian functions in the symmetric form, with the motion of the center of inertia appearing as a separate term

$$H = \frac{P^2}{2M} + \sum_{i=1}^{3}\left[\frac{q_i^2}{2\mu_i} + V_i(\eta_i)\right], \qquad L = \frac{M\mathbf{R}^{\cdot 2}}{2} + \sum_{i=1}^{3}\left[\frac{\mu_i\eta_i^{\cdot 2}}{2} - V_i(\eta_i)\right] \qquad (1.9)$$

When the particles are identical in mass, the effective masses in (1.9) are replaced by $\mu = m/3$, the Lagrangian function is supplemented with undefined Lagrange multipliers $\lambda(\lambda_x, \lambda_y, \lambda_z)$ and the identity (1), and the following expressions in the relative coordinates are obtained for the three-particle equations of motion

$$L^* = L + \lambda(\eta_1 + \eta_2 + \eta_3), \qquad \frac{d}{dt}(\nabla_{\mathbf{R}}\cdot L^*) = 0 \qquad (1.10)$$
$$\frac{d}{dt}(\nabla_{\eta_i}\cdot L) = (\nabla_{\eta_i}L) + (\lambda\cdot\nabla_{\eta_j})(\eta_1 + \eta_2 + \eta_3)$$

After this we use the equations of motion in their explicit form

$$\mathbf{R}^{\cdot\cdot} = 0, \qquad \mu\eta_i^{\cdot\cdot} = -\nabla_{\eta_i}V_i + \lambda \qquad (1.11)$$

and the identity $\eta_1^{\cdot\cdot} + \eta_2^{\cdot\cdot} + \eta_3^{\cdot\cdot} \equiv 0$, to find the Lagrange vector multiplier

$$3\lambda = \nabla_{\eta_1}V_1 + \nabla_{\eta_2}V_2 + \nabla_{\eta_3}V \qquad (1.12)$$

When particles move along stationary trajectories, this multiplier is proportional to the resultant of the relative forces which are equal in magnitude to the forces of pairwise interaction of particles. The two-particle potentials have first order derivatives. Therefore, the identity (1.6) implies that the Lagrange vector multiplier and the resultant of the relative forces are identically equal to zero and this, naturally, does not represent an additional restriction on the initial values of the relative vectors.

For $\lambda \equiv 0$, the three-particle Newton equations decompose

$$\mu\eta_i^{\cdot\cdot} = -\nabla_{\eta_i}V_i \qquad (1.13)$$

while the relations (1) and (1.8) connecting the relative coordinates with impulses are retained, and this enables us to complete the proof which still remains somewhat less simple only for the Hamiltonian function. Indeed, computing the total derivative with respect to time of the three-particle Hamiltonian function (1.9) for $m_k = m$, we obtain

$$\frac{dH}{dt} = \sum_{i,j=1}^{3} [\mathbf{q}_j^{\cdot} (\nabla_{\mathbf{q}_j} H_i) + \boldsymbol{\eta}_j^{\cdot} (\nabla_{\boldsymbol{\eta}_j} H_i)] = 2 \sum_{i=1}^{3} [\mathbf{q}_i^{\cdot} + \nabla_{\boldsymbol{\eta}_i} V_i] \boldsymbol{\eta}_i^{\cdot} \qquad (1.14)$$

where the factor of 2 appears because of the relation connecting the relative coordinates and impulses, we can easily confirm using (1.13) that the Hamiltonian function of a closed system consisting of three particles of identical mass, is a sum of the integrals

$$H = \frac{P^2}{2M} + \sum_{j=1}^{3} H_j(\eta_j, q_j) = \varepsilon_0 + \sum_{j=1}^{3} \varepsilon_j \qquad (1.15)$$

where ε_0 and ε_j are constants. Similarly, for the particles of identical mass we show that the moment of impulse of the closed system is equal to the sum of the integrals

$$\mathbf{M} = \mathbf{R} \times \mathbf{P} + \sum_{j=1}^{3} \boldsymbol{\eta}_j \times \mathbf{q}_j = \mathbf{M}_0 + \sum_{j=1}^{3} \mathbf{M}_j \qquad (1.16)$$

where $\mathbf{M}_0$ and $\mathbf{M}_j$ are constant vectors. If the particle masses are different, the expressions for the relative gradients in terms of the gradients of the independent vectors are not symmetric

$$\nabla_{\eta_1} = \frac{m_1}{M} \nabla_{\mathbf{r}_2} - \frac{m_2}{M} \nabla_{\mathbf{r}_1}, \qquad \nabla_{\eta_2} = \frac{m_2}{M} \nabla_{\mathbf{r}_3} - \frac{m_3}{M} \nabla_{\mathbf{r}_2} \qquad (1.17)$$

$$\nabla_{\eta_3} = \frac{m_3}{M} \nabla_{\mathbf{r}_1} - \frac{m_1}{M} \nabla_{\mathbf{r}_3}$$

and the Lagrange multiplier is not proportional to the resultant of the relative forces

$$\lambda = \frac{m_3}{M} \nabla_{\eta_1} V_1 + \frac{m_1}{M} \nabla_{\eta_2} V_2 + \frac{m_2}{M} \nabla_{\eta_3} V_3 \qquad (1.18)$$

In this case we have

$$(m_1 + m_2 - m_3) \nabla_{\eta_1} V_1 + (-m_1 + m_2 + m_3) \nabla_{\eta_2} V_2 + (m_1 - m_2 + m_3) \nabla_{\eta_3} V_3 \equiv 0 \quad (1.19)$$

therefore no additional integrals are revealed in the motion of the system of particles with different masses.

The changes which the relative vector triangle (1) may undergo during the motion, are classified together with the retained moments of relative impulses. If all three moments (1.16) are equal to zero, then the impulses of the quasi-particles are collinear with the corresponding relative vectors. The quasi-particles move along the relative vectors the length of which varies with time, and the vectors undergo no rotation relative to each other during the motion. Consequently, in this case the triangle (1) remains similar. But if the triangle (1) retains its similarity, then we find that for the gravitational potentials the relative force triangle $\eta_1^{-3}\eta_1 + \eta_2^{-3}\eta_2 + \eta_3^{-3}\eta_3 \equiv 0$ must be equilateral during the motion and we have, for $\mathbf{M}_j = 0$, the Lagrange case [1].

When the moments of the relative impulses are not zero, the resultant of the relative forces remains identically zero and the triangle of relative forces varies dissimilarly from (1). In this case the relative vectors rotate with respect to each other and their lengths vary with time. The triangle (1) does not remain similar during the motion and rotates in space relative to the center of inertia, the latter naturally having no effect on the motion.

2. Systems containing a finite number of particles of identical mass. Internal motion of a multi-partical system can be described using not more than $N = C_A^2$ relative coordinates $\eta_j = r_i - r_h$, provided that these coordinates are connected in triads by the following conditions:

$$\sum_{j=s}^{s+2} \delta_j \eta_j \equiv 0 \qquad (s \leqslant N - 3) \tag{2.1}$$

where $\delta_j = \pm 1$ (depending on the selected directions of the relative vectors). Indeed, in this case we can choose the relative coordinates to be equivalent to (2) and show that the Hamiltonian function of the multi-particle system and its moment of impulse will be sums of the corresponding integrals

$$H = \frac{P^2}{2M} + \sum_{j=1}^{N} \left[\frac{q_j^2}{2\mu} + V_j(\eta_j) \right] = \varepsilon_0 + \sum_{j=1}^{N} \varepsilon_j$$

$$\mathbf{M} = \mathbf{R} \times \mathbf{P} + \sum_{j=1}^{N} \eta_j \times \mathbf{q}_j = M_0 + \sum_{j=1}^{N} \mathbf{M}_j \tag{2.2}$$

in which $M = Am$, $\mu = m/A$, where A is a number of particles.

The additional integrals of relative motion obtained here do not agree with the results of the Bruns theorem [3] which states that the classical integrals are the only independent integrals of the multi-particle motion. Indeed, Bruns based his proof on the equations of motion written in the independent coordinates [3]

$$\frac{d\mathbf{r}_k}{dt} = \nabla_{\mathbf{p}_k} H, \qquad \frac{d\mathbf{p}_k}{dt} = -\nabla_{\mathbf{r}_k} H \tag{2.3}$$

while in the present case we have more equations

$$\frac{d\mathbf{R}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{P}}{dt} = 0; \quad \frac{d\eta_j}{dt} = \nabla_{\mathbf{q}_j} H_j, \quad \frac{d\mathbf{q}_j}{dt} = -\nabla_{\eta_j} H_j \tag{2.4}$$

where $\mathbf{v}$ is the velocity of the center of inertia, and the relative coordinates in these equations are connected by certain conditions. This naturally makes it impossible, according to Bruns, to prove the existence of additional laws of conservation.

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